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Geometric Inequalities with polynomial $2xy + 2yz + 2zx - x^2 - y^2 - z^2$

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ABSTRACT. This paper's aim is to explore the usage of symmetric polynomial

$$\Delta(x, y, z) := 2xy + 2yz + 2zx - x^2 - y^2 - z^2$$

in various geometric inequalities related to triangle.In particular we will show how $\Delta(a, b, c)$,where a, b, c define a triangle, can be used along side of R, r, s to give a new interpretation ((Δ,*r,s*)- *form*) of Hadwiger-Finsler, Blundon's and many others well known and new inequalities. Also we obtain the best quadratic (R, r)-minorant for $\Delta(a, b, c)$ and linear (s, r)-majorant for sum of medians.

1 INTRODUCTION: NOTATIONS AND BASIC CORRELATIONS

Symmetric polynomial

$$\Delta(x, y, z) := 2xy + 2yz + 2zx - x^2 - y^2 - z^2$$

is not a positive definite quadratic form , really:

$$\Delta\left(p + q, \frac{q + r}{2}, \frac{q - r}{2}\right) = q^2 - p^2 - r^2$$

And even requiring $x, y, z > 0$ doesn't guarantee positivity of $\Delta(x, y, z)$. But $\Delta(x, y, z)$ acquires a special meaning for positive x, y, z since in this case inequality $\Delta(x, y, z) > 0$ is equivalent to triangle inequalities for numbers $\sqrt{x}, \sqrt{y}, \sqrt{z}$, that is

$$\Delta(x, y, z) > 0 \iff \sqrt{x} + \sqrt{y} > \sqrt{z}, \sqrt{y} + \sqrt{z} > \sqrt{x}, \sqrt{z} + \sqrt{x} > \sqrt{y}$$

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It is interesting to note that for positive numbers a,b,c, inequality $\Delta\left(a^2,b^2,c^2\right)>0$ characterizes a,b,c as side-lengths of a triangle with area $\dfrac{\sqrt{\Delta\left(a^2,b^2,c^2\right)}}{4}$ and inequality $\Delta\left(a^4,b^4,c^4\right)>0$ characterizes an acute triangle with side-lengths a,b,c. If $\Delta\left(a^n,b^n,c^n\right)>0$ for all natural n then a,b,c represent side-lengths of an isosceles triangle with the lateral side not less than the base [7].
Let a,b,c be side-lengths of a triangle $\triangle ABC$ and let F,s,R, and r be, respectively, area, semiperimeter, circumradius, and inradius of $\triangle ABC$. Also, let r_x be exradius corresponding to a side $x\in\{a,b,c\}$. We shall add

$$\Delta=\Delta(a,b,c)=2ab+2ac+2bc-a^2-b^2-c^2$$

to this list of triangle characteristics.
Using these notations we can write down more representations for $\Delta=\Delta(a,b,c)$:

i $\Delta=\sum\limits_{cyc}\left(a^2-(b-c)^2\right)=2\sum\limits_{cyc}a(s-a)=4\sum\limits_{cyc}(s-a)(s-c)=$
$4s^2-2\left(a^2+b^2+c^2\right)=4(ab+bc+ca)-4s^2$

ii Since $ab+bc+ca=s^2+r(4R+r),\sum\limits_{cyc}r_a=4R+r,\dfrac{r_a}{s}=\tan\dfrac{A}{2}$, we have

$$\Delta=4r(4R+r)=4r\sum_{cyc}r_a=4F\sum_{cyc}\tan\frac{A}{2}.$$

Should also be noted that $\Delta\left(a^2,b^2,c^2\right)=16F^2=16r^2s^2$.
There are many ways to define a triangle. In particular, the most common way is to define a triangle as a triplet (a,b,c) of positive real numbers that satisfy the Triangle Inequalities:

(TI) $\qquad a+b>c, b+c>a, c+a>b$ (*or* $a,b,c<s$)

Triangle,defined in such a way will be denoted by $T(a,b,c)$. Let $x=s-a, y=s-b, z=s-c$ then $a=y+z, b=z+x, c=x+y$, where $x,y,z>0$.Thus, any three positive numbers x,y,z determine a triangle $T(y+z,z+x,x+y)$ and we will call such a representation of a triangle $T(a,b,c)$ a *free parametrization*, because the numbers x,y,z do not depend on each other. In that case $\Delta=\Delta(a,b,c)=\Delta(y+z,z+x,x+y)=4(xy+yz+zx)$.
Let $\mathcal{F}(R,r,s):=4R(R-2r)^3-\left(s^2-2R^2-10Rr+r^2\right)^2$. Note note that three positive numbers R,r,s define a triangle with circumradius R, inradius r and semiperimeter s if and only if the *Fundamental Geometric Inequality* (FGI) $\mathcal{F}(R,r,s)\geq 0$ holds ([1,p.4,inequality(12)] or [2,p.54,Theorem 2]).
This inequality is most commonly used in the form

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R(R-2r)}\leq s^2\leq$$

$$\leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}.$$

Since $\Delta = 4r(4R+r) \iff R = \dfrac{\Delta - 4r^2}{16r}$ then three positive numbers Δ, r and s determine some triangle $T(a,b,c)$ with inradius r, semiperimeter s and $\Delta = \Delta(a,b,c)$ if and only if

$$\mathcal{F}\left(\frac{\Delta - 4r^2}{16r}, r, s\right) \geq 0 \iff 72r^2s^2\Delta + s^2\Delta^2 - \Delta^3 \geq 16r^2s^2\left(27r^2 + 4s^2\right).$$

In general, any inequality in the form $\mathcal{G}(R,r,s) \geq 0$ is equivalent to its (Δ, r, s)-form, obtained by replacing R with $\dfrac{\Delta - 4r^2}{16r}$, namely to inequality

$$\mathcal{G}\left(\frac{\Delta - 4r^2}{16r}, r, s\right) \geq 0.$$

For example, (Δ, r, s)-form of inequalities

$3r(4R+r) \leq s^2, 2r \leq R, s\sqrt{3} \leq 4R + r$

are, respectively,

$3\Delta \leq 4s^2, 36r^2 \leq \Delta, 4\sqrt{3}rs \leq \Delta.$

Remark 1. *Note that inequality* $4\sqrt{3}rs \leq \Delta$ *is a* $(\Delta$,r,s$)$ – form *of Hadwiger-Finsler Inequality*

$$4\sqrt{3}F + (a-b)^2 + (b-c)^2 + (c-a)^2 \leq a^2 + b^2 + c^2 \qquad \textbf{(HF)}$$

Thus, **(HF)** $\iff 4\sqrt{3}F \leq \Delta \iff 4\sqrt{3}rs \leq \Delta \iff s\sqrt{3} \leq 4R + r$.

Since $16Rr = \Delta - 4r^2$ then inequality $16Rr - 5r^2 \leq s^2$ (Gerretsen) can be rewritten in (Δ, r, s)-form

$$\Delta \leq s^2 + 9r^2 \ \textbf{(DG)}$$

and, using inequality $3\sqrt{3}r \leq s$, we obtain:

$$\Delta \leq s^2 + 9r^2 \leq s^2 + \sqrt{3}sr \implies$$

$$\Delta \leq s^2 + \sqrt{3}sr.$$

Since $\Delta = \Delta(y+z, z+x, x+y) = 4(xy+yz+zx)$ and $r^2 = \dfrac{xyz}{x+y+z}$ then, using free parametrization $(a,b,c) = (y+z, z+x, x+y)$,we obtain the following algebraic representations of Hadwiger-Finsler and Gerretsen inequalities:

(HF) $\iff 3 \cdot 16r^2s^2 \leq \Delta^2 \iff xyz(x+y+z) \leq (xy+yz+zx)^2 \iff \sum_{cyc} x^2(y-z)^2 \geq 0;$

(DG) $\iff 4(xy+yz+zx) \leq (x+y+z)^2 + \dfrac{9xyz}{x+y+z} \iff \Delta(x,y,z) \leq \dfrac{9xyz}{x+y+z} \iff$

$9xyz - 4(x+y+z)(xy+yz+zx) + (x+y+z)^3 \geq 0 \iff$

$\sum_{cyc} x(x-y)(x-z) \geq 0$(Schure Inequality).

2 τ AND τ^{-1}– TRANSFORMATIONS

Here we will consider two triangle transformations where Δ plays an important role and which will allow us to obtain new geometric inequalities and establish equivalence of several well-known geometric inequalities.

2.1. τ transformation.

Let $a_\tau = a(s-a), b_\tau = b(s-b), c_\tau = c(s-c)$. Numbers a_τ, b_τ, c_τ are positive and satisfy the triangle inequalities, and therefore determine a triangle $T(a_\tau, b_\tau, c_\tau)$. Indeed, $b_\tau + c_\tau - a_\tau = b(s-b) + c(s-c) - a(s-a) = s(b+c-a) - b^2 - c^2 + a^2 = \dfrac{(b+c)^2 - a^2 + 2a^2 - 2(b^2+c^2)}{2} = \dfrac{a^2 - (b-c)^2}{2} = 2(s-b)(s-c)$ and cyclically we have $c_\tau + a_\tau - b_\tau = 2(s-c)(s-a)$ and $a_\tau + b_\tau - c_\tau = 2(s-a)(s-b)$.
Let $s_\tau, F_\tau, R_\tau, r_\tau$ be semiperimeter, area, circumradius and inradius of the triangle $T(a_\tau, b_\tau, c_\tau)$. Then $s_\tau = \dfrac{a(s-a) + b(s-b) + c(s-c)}{2} = \dfrac{\Delta}{4}$ and since $s_\tau - a_\tau = \dfrac{b_\tau + c_\tau - a_\tau}{2} = (s-b)(s-c)$ and the cyclic $s_\tau - b_\tau = (s-c)(s-a), s_\tau - c_\tau = (s-a)(s-b)$ we obtain

$$F_\tau = \sqrt{s_\tau(s_\tau - a_\tau)(s_\tau - b_\tau)(s_\tau - c_\tau)} = \sqrt{\frac{\Delta}{4}\cdot(s-a)^2(s-b)^2(s-c)^2} = \frac{F^2\sqrt{\Delta}}{2s}$$

and $a_\tau b_\tau c_\tau = abc(s-a)(s-c)(s-a) = \dfrac{4RF^2}{s}$.

Also we obtain $R_\tau = \dfrac{a_\tau b_\tau c_\tau}{4F_\tau} = \dfrac{abcs(s-a)(s-c)(s-a)}{2F^2\sqrt{\Delta}} = \dfrac{abc}{2\sqrt{\Delta}} = R\cdot\dfrac{2F}{\sqrt{\Delta}}$ and

$$r_\tau = \frac{F_\tau}{s_\tau} = \frac{F^2\sqrt{\Delta}}{2s\cdot\frac{\Delta}{4}} = \frac{F}{s}\cdot\frac{2F}{\sqrt{\Delta}} = r\cdot\frac{2F}{\sqrt{\Delta}}.$$

Such transformation of triangle $T(a,b,c)$ we will call τ–transformation. Thus, applying τ–transformation to triangle $T(a,b,c)$ with (s,R,r) we obtain triangle $T(a_\tau, b_\tau, c_\tau)$ with $(s_\tau, R_\tau, r_\tau) = \left(\dfrac{\Delta}{4}, \dfrac{2F}{\sqrt{\Delta}}\cdot R, \dfrac{2F}{\sqrt{\Delta}}\cdot r\right)$ and $F_\tau = \dfrac{F^2\sqrt{\Delta}}{2s}$.
Let $\Delta_\tau := \Delta(a_\tau, b_\tau, c_\tau)$. Then $\Delta_\tau = 4r_\tau(4R_\tau + r_\tau) =$

$$\frac{8rF}{\sqrt{\Delta(a,b,c)}}\left(\frac{8RF}{\sqrt{\Delta(a,b,c)}} + \frac{2rF}{\sqrt{\Delta(a,b,c)}}\right) = \frac{16rF^2(4R+r)}{\Delta(a,b,c)} = 4F^2$$

Since $a_{\tau\tau} = (a_\tau)_\tau = a_\tau(s_\tau - a_\tau) = a(s-a)(s-c)(s-a) = ar^2 s$ then triangle $T(a_{\tau\tau}, b_{\tau\tau}, c_{\tau\tau})$ is similar to triangle $T(a,b,c)$ with coefficient of similarity of $r^2 s$.

2.2. τ^{-1}-transformation.

For any triangle $T(a,b,c)$ lets consider the triangle $T(a_{\tau^{-1}}, b_{\tau^{-1}}, c_{\tau^{-1}})$, where $a_{\tau^{-1}} = \dfrac{r_b + r_c}{\sqrt{s}}, b_{\tau^{-1}} = \dfrac{r_c + r_a}{\sqrt{s}}, c_{\tau^{-1}} = \dfrac{r_a + r_b}{\sqrt{s}}$

Since $a_{\tau-1}=\dfrac{1}{\sqrt{s}}\left(\dfrac{F}{s-b}+\dfrac{F}{s-b}\right)=\dfrac{aF}{(s-b)(s-c)\sqrt{s}}=\dfrac{a(s-a)}{r\sqrt{s}}=\dfrac{a_\tau}{r\sqrt{s}}$ then $(a_\tau)_{\tau-1}=\dfrac{a_{\tau\tau}}{r_\tau\sqrt{s_\tau}}=\dfrac{ar^2s}{\dfrac{2r^2s}{\sqrt{\Delta}}\cdot\sqrt{\dfrac{\Delta}{4}}}=a.$

Thus, $(a_{\tau-1},b_{\tau-1},c_{\tau-1})=\dfrac{1}{r\sqrt{s}}(a_\tau,b_\tau,c_\tau)$ and $((a_\tau)_{\tau-1},(b_\tau)_{\tau-1},(c_\tau)_{\tau-1})=(a,b,c)$ and, therefore, τ^{-1} -transformation is an inverse to τ -transformation.
Due to similarity of triangles $T(a_{\tau-1},b_{\tau-1},c_{\tau-1})$ and $T(a_\tau,b_\tau,c_\tau)$ with coefficient $\dfrac{1}{r\sqrt{s}}=\dfrac{\sqrt{s}}{F}$ we have $s_{\tau-1}=\dfrac{\Delta}{4r\sqrt{s}},F_{\tau-1}=\dfrac{\sqrt{\Delta}}{2},R_{\tau-1}=R\cdot\dfrac{2\sqrt{s}}{\sqrt{\Delta}},r_{\tau-1}=r\cdot\dfrac{2\sqrt{s}}{\sqrt{\Delta}}.$

Theorem 1. Hadwiger-Finsler Inequality $\Delta\geq 4\sqrt{3}F$ is equivalent to inequality $\Delta\leq\frac{4}{3}s^2.$

Proof. Let inequality $\Delta\geq 4\sqrt{3}F$ holds for any $T(a,b,c)$ then, in particular, for $T(a_\tau,b_\tau,c_\tau)$ we have $\Delta_\tau\geq 4\sqrt{3}F_\tau \iff \dfrac{\Delta(a^2,b^2,c^2)}{4}\geq 4\sqrt{3}\cdot\dfrac{F^2\sqrt{\Delta(a,b,c)}}{2s}\iff$ $4F^2\geq 4\sqrt{3}\cdot\dfrac{F^2\sqrt{\Delta(a,b,c)}}{2s}\iff\dfrac{2s}{\sqrt{3}}\geq\sqrt{\Delta(a,b,c)}\iff\dfrac{4}{3}s^2\geq\Delta(a,b,c).$
Assume now that $\frac{4}{3}s^2\geq\Delta(a,b,c)$ holds for any $T(a,b,c)$. Then, in particular $\dfrac{4}{3}s_\tau^2\geq\Delta(a_\tau,b_\tau,c_\tau)\iff\dfrac{4}{3}\cdot\left(\dfrac{\Delta(a,b,c)}{4}\right)^2\geq\dfrac{\Delta(a^2,b^2,c^2)}{4}\iff$ $\Delta^2(a,b,c)\geq 3\Delta(a^2,b^2,c^2)\iff\Delta^2(a,b,c)\geq 3\cdot 16F^2\iff\Delta\geq 4\sqrt{3}F.$
Thus, for any triangle $T(a,b,c)$ holds inequality $4\sqrt{3}rs\leq\Delta\leq\frac{4}{3}s^2$

Remark 2. *Of course inequality $\Delta\leq\frac{4}{3}s^2$ can be proved without $\tau-$transformation. Indeed, since $\Delta=4(ab+bc+ca)-4s^2$ then $4s^2-3\Delta=16s^2-12(ab+bc+ca)=4\left(4s^2-3(ab+bc+ca)\right)=$ $4\left(a^2+b^2+c^2-ab-bc-ca\right)\geq 0$. But the use of $\tau-$transformation gives us one more proof of Hadwiger-Finsler inequality.*

Theorem 2. Inequality $\Delta\sqrt{\Delta}\leq 4abc+8\left(3\sqrt{3}-4\right)(s-a)(s-b)(s-c)$ holds for any triangle $T(a,b,c)$.

Proof. Applying Blundon's inequality $s\leq 2R+\left(3\sqrt{3}-4\right)r$ to triangle $T(a_\tau,b_\tau,c_\tau)$ we obtain $s_\tau\leq 2R_\tau+\left(3\sqrt{3}-4\right)r_\tau\iff\dfrac{\Delta}{4}\leq 2\cdot R\cdot\dfrac{2F}{\sqrt{\Delta}}+\left(3\sqrt{3}-4\right)\cdot r\cdot\dfrac{2F}{\sqrt{\Delta}}\iff$ $\dfrac{\Delta(a,b,c)}{4}\leq\dfrac{4RF}{\sqrt{\Delta}}+\dfrac{2\left(3\sqrt{3}-4\right)rF}{\sqrt{\Delta}}\iff\Delta\sqrt{\Delta}\leq 4abc+8\left(3\sqrt{3}-4\right)r^2s\iff$

$\Delta\sqrt{\Delta} \le 4abc + 8\left(3\sqrt{3}-4\right)(s-a)(s-b)(s-c)$.

Theorem 3. In any triangle $T(a,b,c)$ the following inequalities hold:
1. (a) $64F^2 \le \Delta^2 + 12r^2\Delta$
(b) $\Delta^2 - s^2\Delta \le 12F^2$

Proof. First note that $s^2 \le 4R^2+5Rr+r^2$. (This inequality immediately follows from $s^2 \le 4R^2+4Rr+3r^2$ and $2r \le R$. Or, in a free-parametrization of $T(a,b,c) = T(y+z,z+x,x+y)$ it is equivalent to $\sum\limits_{cyc} z^3(x-y)^2 \ge 0$).Since $\dfrac{R}{r} = \dfrac{\Delta-4r^2}{16r^2}$ and $4R^2+5Rr+r^2 = (4R+r)(R+r) = r\ (4R+r)\left(1+\dfrac{R}{r}\right) = \dfrac{\Delta}{4}\left(1+\dfrac{\Delta-4r^2}{16r^2}\right) = \dfrac{\Delta\left(\Delta+12r^2\right)}{16r^2}$ then $s^2 \le 4R^2+5Rr+r^2 \iff 64F^2 \le \Delta\left(\Delta+12r^2\right)$.
Applying τ–transformation to inequality $64F^2 \le \Delta\left(\Delta+12r^2\right)$ we obtain

$$64F_\tau^2 \le \Delta_\tau\left(\Delta_\tau+12r_\tau^2\right) \iff 64\cdot\left(\frac{F^2\sqrt{\Delta}}{2s}\right)^2 \le 4F^2\left(4F^2+12\left(\frac{2rF}{\sqrt{\Delta}}\right)^2\right) \iff$$

$$\frac{16F^4\Delta}{s^2} \le 4F^2\left(4F^2+12\cdot\frac{4r^2F^2}{\Delta}\right) \iff \frac{\Delta}{s^2} \le 1+\frac{12r^2}{\Delta} \iff \Delta^2-s^2\Delta \le 12F^2.$$

Since $\dfrac{\Delta^2-s^2\Delta}{12} \le F^2 \le \dfrac{\Delta^2+12r^2\Delta}{64}$ then

$$\frac{\Delta^2-s^2\Delta}{12} \le \frac{\Delta^2+12r^2\Delta}{64} \iff 13\Delta \le 16s^2+36r^2 \text{ and we have}$$

$\Delta \le s^2+9r^2 \le \dfrac{16s^2+36r^2}{13}$.On the other hand, inequality $F^2 \le \dfrac{\Delta^2+12\Delta r^2}{64}$ is stronger than $F^2 \le \dfrac{\Delta^2}{48} \iff$ **(HF)**. Indeed, $\dfrac{\Delta^2+12\Delta r^2}{64} \le \dfrac{\Delta^2}{48} \iff$

$$\frac{\Delta^2+12\Delta r^2}{4} \le \frac{\Delta^2}{3} \iff 3\Delta^2+36\Delta r^2 \le 4\Delta^2 \iff 36r^2 \le \Delta.$$

Theorem 4. Inequalities $\Delta \le s^2+9r^2$ and $F^2 \le \dfrac{\Delta^3}{64\left(\Delta-9r^2\right)}$ are equivalent.

Proof. Applying τ-transformation to inequality $\Delta \le s^2+9r^2$ (a *Δ-r-s form of Gerretsen Inequality*) we get $\Delta_\tau \le s_\tau^2+9r_\tau^2 \iff 4F^2 \le \dfrac{\Delta^2}{16}+9r^2\cdot\dfrac{4F^2}{\Delta} \iff$

$$4F^2\left(1-\frac{9r^2}{\Delta}\right) \le \frac{\Delta^2}{16} \iff F^2 \le \frac{\Delta^3}{64\left(\Delta-9r^2\right)}.$$

Remark 2. Using free parametrization $(a,b,c) = (y+z,z+x,x+y)$ we can rewrite inequality $F^2 \le \dfrac{\Delta^3}{64\left(\Delta-9r^2\right)}$ in the form $\sum\limits_{cyc} y^2z^2(x-y)(x-z) \ge 0$.The latter inequality can be obtained from Schure Inequality $\sum_{cyc} x(x-y)(x-z) \geq 0$ by replacing (x,y,z) with (yz,zx,xy).

3 (R,r)- majorants, minorants.

3.1. The family of (R,r)-linear majorant for semiperimeter s

First we will prove

Lemma 1. Let μ and ν be non-negative real numbers.
Inequality $2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)} \leq (\mu R+\nu r)^2$ with equality condition of $R=2r$ holds if and only if $2 \leq \mu \leq \dfrac{3\sqrt{3}}{2}$ and $\nu = 3\sqrt{3}-2\mu$.

Proof. **Necessity**. Let $t=\dfrac{R}{r}$ then we have inequality
$2t^2+10t-1+2(t-2)\sqrt{t(t-2)} \leq (\mu t+\nu)^2$ which holds for any $t \geq 2$ and equality occurs if $t=2$. Then $1 \leq \dfrac{(\mu t+\nu)^2}{2t^2+10t-1+2(t-2)\sqrt{t(t-2)}}$ yields
$1 \leq \lim\limits_{t\to\infty} \dfrac{(\mu t+\nu)^2}{2t^2+10t-1+2(t-2)\sqrt{t(t-2)}} \iff 1 \leq \dfrac{\mu^2}{4} \iff \mu \geq 2.$
For $t=2$ we have $2\cdot 2^2+10\cdot 2-1 = (\mu\cdot 2+\nu)^2 \iff 27 = (2\mu+\nu)^2 \iff$
$2\mu+\nu = 3\sqrt{3} \iff \nu = 3\sqrt{3}-2\mu$ and, therefore, $3\sqrt{3}-2\mu \geq 0 \iff \mu \leq \dfrac{3\sqrt{3}}{2}$.

Sufficiency. Let $\mu \in \left[2, \dfrac{3\sqrt{3}}{2}\right]$. Since $\mu R + \left(3\sqrt{3}-2\mu\right) r = \mu(R-2r)+3\sqrt{3}r$ isn't decreasing in μ and $2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}$
$\leq 4R^2+4Rr+3r^2$ then suffice to prove
$4R^2+4Rr+3r^2 \leq \left(2R\mu+\left(3\sqrt{3}-2\mu\right)r\right)^2$ for $\mu=2$.
We have $\left(2R+\left(3\sqrt{3}-4\right)r\right)^2-\left(4R^2+4Rr+3r^2\right) = 4r\left(3\sqrt{3}-5\right)(R-2r) \geq 0.$
From $s^2 \leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R(R-2r)}$ and the lemma above it follows:

Corollary 1. For any $2 \leq \mu \leq \dfrac{3\sqrt{3}}{2}$ holds inequality $s \leq \mu R + \left(3\sqrt{3}-2\mu\right) r$.

Corollary 2. For any $2 \leq \mu_1 \leq \mu_2 \leq \dfrac{3\sqrt{3}}{2}$ we have
$s \leq \mu_1 R + \left(3\sqrt{3}-2\mu_1\right) r \leq \mu_2 R + \left(3\sqrt{3}-2\mu_2\right) r.$

Proof. $\mu_2 R + \left(3\sqrt{3}-2\mu_2\right) r - \left(\mu_1 R + \left(3\sqrt{3}-2\mu_1\right) r\right) = (\mu_2-\mu_1)(R-2r) \geq 0.$

So, Blundon's Inequality $s \leq 2R+\left(3\sqrt{3}-4\right)r$ (that corresponds to $\mu=2$) gives the best (R,r) −linear majorant for s.

For $\mu=\dfrac{4}{\sqrt{3}}$ *we obtain inequality* $s \leq \dfrac{4}{\sqrt{3}}R+\left(3\sqrt{3}-2\cdot\dfrac{4}{\sqrt{3}}\right)r \iff s \leq \dfrac{1}{\sqrt{3}}(4R+r) \iff \sqrt{3}s \leq 4R+r \iff 4\sqrt{3}F \leq \Delta.$

For $\mu=\dfrac{3\sqrt{3}}{2}$ we obtain inequality $s \leq \dfrac{3\sqrt{3}}{2}R$ and for $\mu=\dfrac{2\sqrt{3}+3}{3} \in \left[2,\dfrac{3\sqrt{3}}{2}\right]$ we obtain $s \leq \dfrac{2\sqrt{3}+3}{3}R+\left(3\sqrt{3}-2\cdot\dfrac{2\sqrt{3}+3}{3}\right)r \iff s \leq \left(\dfrac{2}{\sqrt{3}}+1\right)R+\dfrac{5\sqrt{3}-6}{3}r.$

Since $2<\dfrac{2\sqrt{3}+3}{3}<\dfrac{4}{\sqrt{3}}<\dfrac{3\sqrt{3}}{2}$ then

$$s \leq 2R+\left(3\sqrt{3}-4\right)r \leq \left(\frac{2}{\sqrt{3}}+1\right)R+\frac{5\sqrt{3}-6}{3}r \leq \frac{1}{\sqrt{3}}(4R+r) \leq \frac{3\sqrt{3}}{2}R.$$

Inequalities $s \leq 2R+\left(3\sqrt{3}-4\right)r$, $s \leq \dfrac{1}{\sqrt{3}}(4R+r)$, $s \leq \dfrac{3\sqrt{3}}{2}R$ are well known, but what is so special about $\mu=\dfrac{2\sqrt{3}+3}{3}$ that we must pay attention to it?
The answer became obvious after considering the linear (r,s) majorant for sum of medians. But first lets look at the (r,s)-quadratic minorants for Δ.

3.2. Quadratic (r,s)-minorants for Δ

By substitution of $R=\dfrac{\Delta-4r^2}{16r}$ in inequality
$s \leq \mu R+\left(3\sqrt{3}-2\mu\right), \mu \in \left[2,\dfrac{3\sqrt{3}}{2}\right]$ we obtain

$$s \leq \mu\cdot\frac{\Delta-4r^2}{16r}+\left(3\sqrt{3}-2\mu\right)r \iff 16rs \leq \mu\Delta+\left(48\sqrt{3}-36\mu\right)r^2 \iff$$

1. $\alpha rs-\beta r^2 \leq \Delta$, where $\alpha:=\dfrac{16}{\mu}$ and $\beta:=\dfrac{12\left(4\sqrt{3}-3\mu\right)}{\mu}$

In particular, if $\mu=2,\ \dfrac{2\sqrt{3}+3}{3},\ \dfrac{4}{\sqrt{3}},\ \dfrac{3\sqrt{3}}{2}$ we obtain respectively:

2. $8rs-12\left(2\sqrt{3}-3\right)r^2 \leq \Delta,$
3. $16\left(2\sqrt{3}-3\right)sr-36\left(2-\sqrt{3}\right)^2r^2 \leq \Delta,$
4. $4\sqrt{3}rs \leq \Delta \iff$ **(HF)**
5. $\dfrac{32\sqrt{3}}{9}rs+4r^2 \leq \Delta.$

Since $s \geq 3\sqrt{3}r$ then $\alpha rs-\beta r^2=\dfrac{16r\left(s-3\sqrt{3}r\right)}{\mu}+36r^2$ is decreasing in

$\mu \in \left[2, \frac{3\sqrt{3}}{2}\right]$ and, therefore, $\mu = 2$ give us *the best (r,s)-quadratic minorants* for Δ

6. $8rs - 12\left(2\sqrt{3} - 3\right)r^2 \le \Delta$

that is

$$\frac{16rs}{\mu} - \frac{12\left(4\sqrt{3} - 3\mu\right)}{\mu}r^2 \le 8rs - 12\left(2\sqrt{3} - 3\right)r^2 \le \Delta, \ \mu \in \left[2, \frac{3\sqrt{3}}{2}\right].$$

In particular, since $2 < \dfrac{2\sqrt{3}+3}{3} < \dfrac{4}{\sqrt{3}} < \dfrac{3\sqrt{3}}{2}$ we have the chain of inequalities:

$$\frac{32\sqrt{3}}{9}rs + 4r^2 \le 4\sqrt{3}rs \le 16\left(2\sqrt{3} - 3\right)sr - 36\left(2 - \sqrt{3}\right)^2 r^2 \le$$
$$8rs - 12\left(2\sqrt{3} - 3\right)r^2 \le \Delta.$$

(Inequality **(6)** can be considered as refinement of Hadwiger-Finsler Inequality in the (Δ, r, s)- form and it is analogous to the Blundon's Inequality which give the best linear (R, r) majorant to s).

3.3. Linear (r,s)-majorant for sum of medians.

Lemma 2. Let m_a and m_b be medians of a triangle with side-lengths a, b, c. Then

$$m_a m_b \le \frac{2c^2 + ab}{4}.$$

Proof. Since $m_a^2 = \dfrac{2\left(b^2 + c^2\right) - a^2}{4}, m_b^2 = \dfrac{2\left(c^2 + a^2\right) - b^2}{4}$

then $16\left(\left(\dfrac{2c^2 + ab}{4}\right)^2 - m_a^2 m_b^2\right) = \left(2\left(b^2 + c^2\right) - a^2\right)\left(2\left(c^2 + a^2\right) - b^2\right) -$

$\left(2c^2 + ab\right)^2 = 2\left(\left(a^2 - b^2\right)^2 - c^2\left(a - b\right)^2\right) = 2\left(a - b\right)^2\left(a + b + c\right)\left(a + b - c\right) \ge 0.$

Corollary 3. $\left(m_a + m_b + m_c\right)^2 \le \dfrac{16s^2 - 3\Delta}{4}.$

Proof. Since

$m_a m_b + m_b m_c + m_c m_a \le \sum\limits_{cyc} \dfrac{2c^2 + ab}{4} = \dfrac{2\left(a^2 + b^2 + c^2\right) + ab + bc + ca}{4}$ and for any k and l holds

identity $k\left(a^2 + b^2 + c^2\right) + l\left(ab + bc + ca\right) = \left(l + 2k\right)s^2 - \dfrac{\left(2k - l\right)\Delta}{4}$ then

$\left(m_a + m_b + m_c\right)^2 = m_a^2 + m_b^2 + m_c^2 + 2\left(m_a m_b + m_b m_c + m_c m_a\right) = \dfrac{3}{4}\left(a^2 + b^2 + c^2\right) +$

$$2(m_am_b+m_bm_c+m_cm_a)\le\frac{7(a^2+b^2+c^2)+2(ab+bc+ca)}{4}=\frac{16s^2-3\Delta}{4}$$

Theorem 5. Let m_a,m_b,m_c be medians of a triangle with semiperimineter s and inradius r. Then

(M) $$m_a+m_b+m_c\le 2s-3\left(2\sqrt{3}-3\right)r.$$ [8]

Proof. By Corollary 3.3.1. we have $m_a+m_b+m_c\le\dfrac{\sqrt{16s^2-3\Delta}}{2}$ and using inequality $\alpha rs-\beta r^2\le\Delta$, where $\alpha:=\dfrac{16}{\mu}$ and $\beta:=\dfrac{12\left(4\sqrt{3}-3\mu\right)}{\mu}$ we obtain:
$16s^2-3\Delta\le16s^2-3\left(\alpha sr-\beta r^2\right)=16s^2-3\alpha sr+3\beta r^2$.Therefore,
$m_a+m_b+m_c\le\dfrac{1}{2}\sqrt{16s^2-3\alpha sr+3\beta r^2}$.Since
$16s^2-3\alpha sr+3\beta r^2=\left(4s-\dfrac{3\alpha r}{8}\right)^2+3r^2\left(\beta-\dfrac{3\alpha^2}{64}\right)$ then
$16s^2-3\alpha sr+3\beta r^2$ becomes a perfect square if and only if
$\beta=\dfrac{3\alpha^2}{64}\iff\dfrac{12\left(4\sqrt{3}-3\mu\right)}{\mu}=\dfrac{3}{64}\cdot\dfrac{256}{\mu^2}\iff4\sqrt{3}-3\mu=\dfrac{1}{\mu}\iff$
$3\mu^2-4\sqrt{3}\mu+1=0\iff\mu=\dfrac{2\sqrt{3}+3}{3}$. For $\mu=\dfrac{2\sqrt{3}+3}{3}$ we have
$\alpha=16\left(2\sqrt{3}-3\right),\dfrac{1}{2}\left(4s-\dfrac{3\alpha r}{8}\right)=2s-\dfrac{3}{16}\cdot16\left(2\sqrt{3}-3\right)r=2s-3\left(2\sqrt{3}-3\right)r$ and, therefore,

$$m_a+m_b+m_c\le2s-3\left(2\sqrt{3}-3\right)r.$$

Thus, only $\mu=\dfrac{2\sqrt{3}+3}{3}$ provide the linear (s,r) –majorant for sum of medians.
Inequality **(M)**, when it already established, can be proven by a shorter way:
Since $ab+bc+ca=s^2+r(4R+r)$ and $a^2+b^2+c^2=2\left(s^2-r(4R+r)\right)$ then
$(m_a+m_b+m_c)^2\le\dfrac{7\left(a^2+b^2+c^2\right)+2(ab+bc+ca)}{4}=$
$\dfrac{14\left(s^2-r(4R+r)\right)+2\left(s^2+r(4R+r)\right)}{4}=4s^2-3r^2-12Rr$ and, therefore, it suffices to prove $4s^2-3r^2-12Rr\le\left(2s-3\left(2\sqrt{3}-3\right)r\right)^2$.
Since $s\le\dfrac{3\sqrt{3}}{2}R$ we have$\left(2s-3\left(2\sqrt{3}-3\right)r\right)^2-\left(4s^2-3r^2-12Rr\right)=$
$12r\left(R-s\left(2\sqrt{3}-3\right)-r\left(9\sqrt{3}-16\right)\right)\ge$
$12r\left(R-\dfrac{3\sqrt{3}}{2}R\cdot\left(2\sqrt{3}-3\right)-r\left(9\sqrt{3}-16\right)\right)=6\left(9\sqrt{3}-16\right)r(R-2r)\ge0.$
(Inequality **(M)** as a conjecture was proposed by Konstantin Knop in private communication).

4 More inequalities with Δ

In conclusion, we will consider a few inequalities with Δ. First we will present a chain of inequalities with Δ (some of them are already well known).

Inequality 1.

Let a, b, c be lengths of sides of a triangle. Then

$$\frac{9a^2b^2c^2}{a^2b^2+b^2c^2+c^2a^2} \leq \frac{3abc(a+b+c)}{a^2+b^2+c^2} \leq \Delta \leq \frac{8abc(ab+bc+ca)}{(a+b)(b+c)(c+a)} \leq \frac{9abc}{a+b+c} \leq \min\left\{\frac{3abc(a+b+c)}{ab+bc+ca}, 3\sqrt[3]{a^2b^2c^2}\right\}.$$

Proof. Since a, b, c are positive, we have

$\frac{9abc}{a+b+c} \leq 3\sqrt[3]{a^2b^2c^2} \iff 3\sqrt[3]{abc} \leq a+b+c$ and

$\frac{9abc}{a+b+c} \leq \frac{3abc(a+b+c)}{ab+bc+ca} \iff 3(ab+bc+ca) \leq (a+b+c)^2$ and

$\frac{8abc(ab+bc+ca)}{(a+b)(b+c)(c+a)} \leq \frac{9abc}{a+b+c} \iff 8(a+b+c)(ab+bc+ca) \leq 9(a+b)(b+c)(c+a)$

(as a side note: all of these inequalities hold for any positive a, b, c).
So it remains to prove:

1. $\frac{3abc(a+b+c)}{a^2+b^2+c^2} \leq \Delta \leq \frac{8abc(ab+bc+ca)}{(a+b)(b+c)(c+a)}$ and

2. $\frac{9a^2b^2c^2}{a^2b^2+b^2c^2+c^2a^2} \leq \frac{3abc(a+b+c)}{a^2+b^2+c^2} \iff$
$$3abc\left(a^2+b^2+c^2\right) \leq (a+b+c)\left(a^2b^2+b^2c^2+c^2a^2\right).$$

Proof. (Inequalities (1) and (2)).

Using free parametrization of the triangle namely, $(a, b, c) = (y+z, z+x, x+y)$,

denoting $p := xy+yz+zx, q := xyz$, and due to homogeneity of the inequalities, assuming $x+y+z=1$, we obtain $s=1, \Delta=4p, abc=p-q, a^2+b^2+c^2=2(1-p), a^2b^2+b^2c^2+c^2a^2=(1-p)^2+4q, (a+b)(b+c)(c+a)=2+p+q$ and inequalities (1) and (2) becomes, respectively,

$$\frac{3(p-q)}{1-p} \leq 4p \leq \frac{8(p-q)(1+p)}{2+p+q} \text{ and}$$

$$3(p-q)(1-p) \leq (1-p)^2+4q \iff (7-3p)q-(4p-1)(1-p) \geq 0.$$

First, lets prove inequality

$\frac{3(p-q)}{1-p} \leq 4p \iff \frac{3(p-q)}{1-p} \leq 4p \iff 3q+p-4p^2 \geq 0$. Since $p = xy+yz+$

$zx \le \frac{(x+y+z)^2}{3} = \frac{1}{3}$ and $\sum_{cyclic} x(x-y)(x-z) \ge 0 \iff q \ge \frac{4p-1}{9}$ then $0 < p \le \frac{1}{3}, q \ge \max\left\{0, \frac{4p-1}{9}\right\}$.For $0 < p \le \frac{1}{4}$ we have $3q+p-4p^2 \ge p(1-4p) \ge 0$ and for $\frac{1}{4} < p \le \frac{1}{3}$we have $3q+p-4p^2 \ge \frac{4p-1}{3} + p - 4p^2 = \frac{(4p-1)(1-3p)}{3} \ge 0.$

Now, lets prove inequality $4p \le \frac{8(p-q)(1+p)}{2+p+q} \iff p \le \frac{2(p-q)(1+p)}{2+p+q} \iff p^2 - (3p+2)q \ge 0$.Since $q = xyz(x+y+z) \le \frac{(xy+yz+zx)^2}{3} = \frac{p^2}{3}$ then $p^2 - (3p+2)q \ge p^2 - (3p+2)\frac{p^2}{3} = \frac{1}{3}p^2(1-3p) \ge 0.$

Thus, it remains to prove inequality $(7-3p)q - (4p-1)(1-p) \ge 0.$

Note that $q \ge \max\left\{0, \frac{(1-p)(4p-1)}{6}\right\}$ since $\sum_{cyclic} x^2(x-y)(x-z) \ge 0 \iff q \ge \frac{(1-p)(4p-1)}{6}$.For $0 < p \le \frac{1}{4}$ we have$(7-3p)q - (4p-1)(1-p) = (7-3p)q + (1-4p)(1-p) \ge (1-4p)(1-p) \ge 0$ and for $\frac{1}{4} < p \le \frac{1}{3}$ we have $(7-3p)q - (4p-1)(1-p) \ge (7-3p)\frac{(1-p)(4p-1)}{6} - (4p-1)(1-p) = \frac{(1-p)(1-3p)(4p-1)}{6} \ge 0$

Remark 4. *In reality, inequality $\Delta \le 3\sqrt[3]{a^2b^2c^2}$ holds for any real a,b,c and it is $\Delta-$ form (up to replacement (a,b,c) with (x^3,y^3,z^3)) of well known [5],[6] inequality*
$(x^3+y^3+z^3)^2 + 3(xyz)^2 \ge 4(x^3y^3+y^3z^3+z^3x^3)$, where $x,y,z \in \mathbb{R}$.

Inequality 2. Let a,b,c be lengths of sides of a triangle. Then for any real positive x,y,z holds inequality:

1. (a) $\frac{xbc}{y+z} + \frac{yca}{z+x} + \frac{xbc}{x+y} \ge \frac{\Delta}{2}$;

   (b) $\frac{xa^2}{y+z} + \frac{yb^2}{z+x} + \frac{zc^2}{x+y} \ge \frac{\Delta}{2}$.

Remark 5. *Inequality* ***(a)*** *is a geometric version of algebraic inequality, proved by* ***M.S.Klamkin*** *for any positive a,b,c,x,y,z and presented as* ***Inequality 1*** *in [1],* ***p.33*** *without proof and with reference to original article. Inequality in* ***(b)*** *is also a geometric version of algebraic* *inequality proved by* **D.S.Mitrinovic,J.E. Pecaric** *for any positive* a, b, c *and real* x, y, z *such that* $x+y, y+z, z+x>0$ *and presented as* **Inequalities 6 and 10** *in [1],* **p.34** *with easy proof.*

So, we will prove only the inequality **(a)** in the form $\sum_{cyc} \frac{x}{a(y+z)} \geq \frac{\Delta}{2abc}$.

Proof. Applying Cauchy Inequality to triples $\left(\frac{x}{\sqrt{ax(y+z)}}, \frac{y}{\sqrt{bx(z+x)}}, \frac{z}{\sqrt{cz(x+y)}}\right)$ and $\left(\sqrt{ax(y+z)}, \sqrt{bx(z+x)}, \sqrt{cz(x+y)}\right)$ we obtain

$$\sum_{cyc} ax(y+z) \cdot \sum_{cyc} \frac{x^2}{ax(y+z)} \geq (x+y+z)^2 \iff \sum_{cyc} \frac{x}{a(y+z)} \geq \frac{(x+y+z)^2}{\sum_{cyc} ax(y+z)}.$$

Since $\frac{(x+y+z)^2}{\sum_{cyc} ax(y+z)} \geq \frac{4(ab+bc+ca)}{(a+b)(b+c)(c+a)}$ (inequality **(D)**, [4]) and $\frac{8(ab+bc+ca)abc}{(a+b)(b+c)(c+a)} \geq \Delta$ we obtain $\sum_{cyc} \frac{x}{a(y+z)} \geq \frac{\Delta}{2abc}$.

Inequality 3. Let $a, b,$and c be lengths of sides of a triangle ABC and P be any point in the triangle. Let d_a, d_b, d_c be distances from point P to sides a, b, c respectively. Then

$$d_a d_b + d_b d_c + d_c d_a \leq \frac{4F^2}{\Delta} = \frac{\Delta(a^2, b^2, c^2)}{4\Delta(a, b, c)}.$$

Proof. Since $ad_a + bd_b + cd_c = 2F$ then by replacing (x, y, z) and (α, β, γ) in inequality $\alpha yz + \beta zx + \gamma xy \leq \frac{\alpha\beta\gamma(x+y+z)^2}{\Delta(\alpha, \beta, \gamma)}$ (inequality **(C)**, [4]) with (ad_a, bd_b, cd_c) and (a, b, c), respectively, we obtain

Remark 6. *This inequality was proven in [4],***p.460** *as inequality* **(DP)** *and originally represented as maximization problem and since* $4d_a d_b + d_b d_c + d_c d_a = \Delta(d_a + d_b, d_b + d_c, d_c + d_a)$ *it can be rewritten as*

$$\Delta(d_a + d_b, d_b + d_c, d_c + d_a)\,\Delta(a, b, c) \leq \Delta(a^2, b^2, c^2).$$

Inequality 4. Let a, b, c be sidelengths of an acute triangle with circumradius R, inradius r and semiperimeter s.Then $\Delta \leq \frac{8rs\sqrt{3} + 4s^2}{5}$.
(This inequality is Δ-r-s *representation of* inequality in **Theorem 1.4** in **[3]**).

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